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Simultaneous Approximation of a Set

CHARLES B. DUNHAM

Computer Science Department, University of Western Ontario, London, Ontario, Canada N6A 5B9

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A family in a linear space is to be simultaneously approximated by a finite-dimensional linear subspace with respect to a norm. Existence of a best approximation is established. Characterization and uniqueness results are obtained. Discretization of the family can be employed to get a best approximation.

Let $\mathscr{F}$ be a linear space with norm $\| \ \|$. Let $\{\phi_1, ..., \phi_n\}$ be a linearly independent subset of $\mathscr{F}$ and define

$$L(A) = \sum_{k=1}^{n} a_k \phi_k .$$

The problem of approximation is: For a given nonempty subset F of $\mathscr{F}$, find a parameter A to minimize

$$e(F, A) = \sup\{\| f - L(A) \| : f \in F\}.$$

Such a parameter A is called best and $L(A)$ is called a best approximation to F. This is a generalization of the classical problem of linear approximation with respect to a norm, in which F consists of a single element f.

The omitted proofs of results in this paper use arguments similar to the classical ones for the case of F a single element.

One of the first to consider this general approximation problem was Golomb [7, pp. 264ff]. It has also been considered by Laurent and Tuan [8] and for F a pair by Goel *et al.* [6]. A related approximation problem is that of approximation by all of $\mathscr{F}$, the problem of centers [9].

A case of special interest has been the case in which $\| \ \|$ is the Chebyshev norm. The first widely distributed result on this was by the author [5]. Laurent and Tuan cite two neglected earlier papers by Golomb and Remez. Diaz and McLaughlin extended the author's results in [2] and studied complex linear Chebyshev approximation in [3]. Nonlinear simultaneous Chebyshev approximation has been studied by Blatt [1].

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The problem of simultaneous approximation arises in approximation of a function (weakly) dependent on a parameter or parameters. For example, let h be a function on $X \times Y$. Let us define

$$g_y(x) = h(x, y), \qquad x \in X, \quad y \in Y,$$

and set $F = \{g_y: y \in Y\}$. In the case that h is measured experimentally and the value of the parameter y is difficult to measure, we may know the family F much better than h. In the case h is approximated by a function of x only, it seems appropriate to approximate the family F simultaneously. In the case that the value of the parameter when the measurement was taken is unknown, we have effectively a multivalued function g of x, and simultaneous approximation is the only possible approach.

This approach is particularly suitable for Chebyshev approximation of a function of several variables by functions of fewer variables. Suppose we wish to approximate h, a function of $X \times Y$ by a function of X. Let s be an approximant and F be as in the preceding paragraph, then

$$\|h - s\|_{\infty} = \sup\{\|f - s\|_{\infty} : f \in F\}.$$

Motivation for studying this problem is also given in [7, pp. 264 ff; 8, p. 138; 9, p. 283].

It can be shown that $e(F, A) = e(\bar{F}, A)$, where $\bar{F}$ is the closure of F. Hence there is no loss of generality in considering F closed.

THEOREM. *Let F be a bounded nonempty set, then there exists a best approximation to F.*

THEOREM. *The set of best coefficient vectors is closed, bounded, and convex.*

COROLLARY. *$e(F, \cdot)$ is a convex function.*

Use of a routine for minimizing a convex function is a general approach to computing best approximations. In some cases this is the only known approach.

Let $M(F, A) = \{f: \|f - L(A)\| = e(F, A)\}$.

THEOREM. *Let F be compact. A necessary and sufficient condition for A to be best to F is that A is best to $M(F, A)$.*

Proof. Sufficiency is obvious. Suppose A is not best to $M(F, A)$. Then there is a coefficient vector B such that $e(M(F, A)), B) < e(M(F, A), A)$. Hence

$$\|f - L(B)\| < \|f - L(A)\|, \qquad f \in M(F, A).$$

By continuity there is an open cover U of $M(F, A)$ such that

(i) $\|f - L(B)\| < \|f - L(A)\|, \quad f \in U$

(ii) for $0 < \lambda \leqslant 1$ and such f,

$$\begin{aligned} \|f - L(\lambda B + (1-\lambda)A)\| &\leqslant \lambda \|f - L(B)\| + (1-\lambda)\|f - L(A)\| \\ &< \|f - L(A)\|. \end{aligned} \tag{1}$$

If U contains all of F, A is not best to F. Otherwise let $V = F \sim U$ be nonempty. As U is open, V is a closed set. Let

$$\eta = \sup\{\|f - L(A)\| : f \in V\},$$

then as V is compact, there is $f \in V$ for which the supremum is attained and since $M(F, A) \cap V$ is empty, $\eta < e(F, A)$. There exists $\mu > 0$ such that $u \| L(B - A)\| < e(F, A) - \eta$. We have for $f \in V$, $0 < \lambda \leqslant \mu$,

$$\begin{aligned} \|f - L(\lambda B + (1-\lambda)A)\| &< \|f - L(A)\| + \|\lambda L(B - A)\| \\ &< \eta + e(F, A) - \eta = e(F, A). \end{aligned}$$

Combining this with (1) we have for all λ in $(0, \min\{\mu, 1\}]$

$$\|f - L(\lambda B + (1-\lambda)A)\| < e(F, A), \qquad f \in F,$$

and by compactness, $e(F, \lambda B + (1-\lambda)A) < e(F, A)$, proving the theorem.

It is a consequence of the theorem that if $M(F, A)$ consists of a single element f and $L(A)$ is a unique best approximation to f, $L(A)$ is a unique best approximation to F.

THEOREM. *Let $\| \; \|$ be strict and F compact. Then a best approximation is unique.*

Let us now consider some special norms. The L_p norms are strict for $1 < p < \infty$ and are covered by the above theorem. The Chebyshev norm is not strict and nonuniqueness can occur. The papers [1; 5] should be consulted. In the case of the L_1 norm, it appears that nonuniqueness can occur easily.

THEOREM. *Let $\int$ be the integral on X. Let $F = \{f_1, f_2\}$, $f_1 \leqslant f_2$. If $f_1 \leqslant L(A) \leqslant f_2$, and $\int (L(A) - f_1) = \int (f_2 - L(A)) = \int (f_2 - f_1)/2$, then $L(A)$ is a best L_1 approximation to F.*

In the case where f_1 and f_2 are different constants and L is a polynomial approximating function of degree $\geqslant 1$, there are many such A's. Carroll [4] gives a sufficient condition for best simultaneous L_1 approximations to be unique.

Let

$$d(F_1, F_2) = \sup\{\inf \|f_1 - f_2\| : f_2 \in F_2\} : f_1 \in F_1\},$$
$$\operatorname{dist}(F_1, F_2) = \max\{d(F_1, F_2)\ d(F_2, F_1)\}.$$

We say $\{F_k\} \to F$ if $\operatorname{dist}(F, F_k) \to 0$.

THEOREM. *Let $\{F_k\} \to F$ and A^k be best to F_k. Then $\{A^k\}$ has an accumulation point and any such accumulation point is best to F.*

A case of particular interest is where F_k is a sequence of subsets of F such that for all $f \in F$, there is a sequence $\{f_k\} \to f$, $f_k \in F_k$. In this case it is seen that $\{F_k\} \to F$ and the theorem can be applied. The case where $\{F_k\}$ is a sequence of finite sets is of practical interest, as computing a best approximation relative to a set F of functions is easier if F is finite. The process of replacing approximation of infinite F by approximation of finite F_k is a process of *discretization*, which is used extensively in many other approximation problems.

REFERENCES

1. H. P. BLATT, "Nicht-Lineare Gleichmässige Simultanapproximation," Dissertation, Universität des Saarlandes, Saarbrücken, 1970.
2. J. B. DIAZ AND H. W. MCLAUGHLIN, Simultaneous approximation of a set of bounded real functions, *Math. Comp.* **23** (1969), 583–593.
3. J. B. DIAZ AND H. W. MCLAUGHLIN, Simultaneous Chebyshev approximation of a set of bounded complex-valued functions, *J. Approximation Theory* **2** (1969), 419–432.
4. M. P. CARROLL, Simultaneous L_1 approximation of a compact set of real-valued functions, *Numer. Math.* **19** (1972), 110–115.
5. C. B. DUNHAM, Simultaneous Chebyshev approximation of functions on an interval, *Proc. Amer. Math. Soc.* **18** (1967), 472–477.
6. D. S. GOEL, A. HOLLAND, C. NASIM, AND B. N. SAHNEY, On best simultaneous approximation in normed linear spaces, *Canad. Math. Bull.* **17** (1974), 523–527.
7. M. GOLOMB, Lectures on Theory of Approximation, Argonne National Laboratory, 1962.
8. P. J. LAURENT AND P. D. TUAN, Global approximation of a compact set by elements of a convex set in a normed space, *Numer. Math.* **15** (1970), 137–150.
9. J. D. WARD, Chebyshev centers in spaces of continuous functions, *Pacific J. Math.* **52** (1974), 283–287.
10. F. DEUTSCH, review of 6, #13556, *Math. Reviews* **51** (1976), 1904.
11. A. S. HOLLAND, B. N. SAHNEY, AND J. TZIMBALARIO, On best simultaneous approximation, *J. Approximation Theory* **17** (1976), 187–188.
12. A. BRONDSTED, A note on best simultaneous approximation in normed linear spaces, *Canad. Math. Bull.* **19** (1976), 359–360.